Endogenous income taxes in OLG economies

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Abstract

This paper introduces endogenous income tax rates as in Schmitt-Grohe and Uribe (1997), into the overlapping generations model with endogenous labor and consumption in both periods of life (e.g., Cazzavillan and Pintus, 2004). It shows that local indeterminacy can occur with small distortionary taxes, provided that the elasticity of capital-labor substitution is less than the share of capital in total income, the wage elasticity of the labor supply is large enough, and the fraction of young-age consumption out of wage income is small enough. This is in contrast with the result of Schmitt-Grohe and Uribe (1997), who show that endogenous income tax rate may itself generate indeterminacy in a standard neoclassical growth model. More important is the fact that increasing the size of tax distortions enlarges the range of values of the consumption-to-wage ratio associated with local indeterminacy if constant government expenditure is financed through labor income taxes, while increasing the size of tax distortions can make shrink the range of values

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of the consumption–to–wage ratio associated with local indeterminacy if constant government expenditure is financed through capital income taxes.

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JEL: C62; E32.

1. Introduction

In this paper, we consider a two-periods overlapping generations model with endogenous labor, consumption in both periods of life and endogenous income taxes. Cazzavillan and Pintus (2004) have pointed out that intertemporal substitution in consumption is a critical element to make expectation-driven fluctuations disappear in the context of OLG economies if the ratio between savings and wage is reasonably low. We show that this element can not be weakened by the presence of endogenous income taxes, as in Schmitt-Grohe and Uribe (1997), since it enables agents to arbitrage away endogenous fluctuations when the ratio between savings and wage is low. We prove that with tax distortions and numerical calibrations for the fundamentals, indeterminacy can occur in a CES economy provided that the elasticity of the labor supply is large enough, the elasticity of the input substitution is less than the share of capital in total income, and the propensity to save out of the wage income takes unrealistic values. Our results are in contrast with what has been shown in Schmitt-Grohe and Uribe (1997): for reasonable values of the model parameters, endogenous income tax rate may itself generate indeterminacy in a standard neoclassical growth model.

Since Reichlin (1986), the Diamond (1965) one-sector overlapping generations model augmented to include endogenous labor supply, external effects and/or fiscal policy has become a popular framework to analyze expectations driven business cycles.1 Unlike those early works that focus on a particular case without first period consumption, recent works such as Cazzavillan and Pintus (2004,

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1For example, Cazzavillan (2001) and Gokan (2009a, 2009b).
2006) and Lloyd-Braga et al. (2007), consider a life-cycle utility function which is first, separable between consumption and leisure, and second, homogenous with respect to young and old consumptions. The main contribution of these papers is to analyze the relationship between external effects and indeterminacy in an aggregate OLG model with consumption in both periods of life. Our paper instead discusses the relationship between fiscal policy and indeterminacy in the very same aggregate OLG model. Particularly, we consider the case where positive government expenditure is financed by labor or capital income taxes in Cazzavillan and Pintus (2004). And we find that compared with Cazzavillan and Pintus (2004) without the government sector, endogenous fluctuations are (un)likely to arise at a wide range of values of the consumption to wage ratio when government expenditure is financed through labor (capital) income taxes. In other words, increasing the size of tax distortions enlarges the range of values of the consumption–to–wage ratio associated with local indeterminacy if constant government expenditure is financed through labor income taxes, while increasing the size of tax distortions can make shrink the range of values of the consumption–to–wage ratio associated with local indeterminacy if constant government expenditure is financed through capital income taxes.

In order to make our analysis tractable while introducing constant government expenditures financed by labor or capital income taxes, together with current consumption, we need two important assumptions on preferences: we assume that a life-cycle utility function is separable between consumption and leisure, and homogenous with respect to consumptions.

Considering a small share of total consumption over the output (less than averages for the OECD countries), we first prove that local indeterminacy of equilibria can be generated when there are small labor income tax rates. Second, we prove that for a given technology (θ in Cazzavillan and Pintus 2004), adding labor income tax rates will make the ratio of consumption while young to saving smaller, thus making sunspots more likely to occur. Finally, we prove that for a given technology, adding labor income tax rates can make larger the upper bound on the elasticity of the input.
substitution associated with multiple equilibria ($\sigma_H$ in Cazzavillan and Pintus 2004), although this bound is still less than the share of capital in total income. To summarize, we show that endogenous labor income taxes are helpful to local indeterminacy, and the indeterminacy conditions in our model are very similar to those obtained in Cazzavillan and Pintus (2004) except that the critical values of the independent parameter ($\sigma$) and the bifurcation parameter ($R_2$) are different from those in their model.

Another important result is that endogenous capital income taxes are not favorable to local indeterminacy. We investigate how government expenditure financed by capital income taxes influences local dynamics near the (normalized) steady state in the same OLG model. And we find that, first, for a small share of total consumption over the output, local indeterminacy can occur when there are small capital income taxes. Second, increasing the size of capital income taxes can make shrink the range of values of the consumption-to-wage ratio associated with local indeterminacy. Lastly, we show that for a given technology ($\theta$), adding capital income taxes can make decrease the critical value of the input substitution ($\sigma$) associated with multiple equilibria. Therefore, endogenous capital income taxes are not favorable to local indeterminacy.

The following intuitive interpretation may help the reader to understand the mechanisms behind these results. Endogenous fluctuations arise due to the interaction of two conflicting effects: when the capital stock increases, it leads to an increase in wage rate and, therefore, an increase in savings which leads the capital stock in the next period to be higher. However, capital accumulation is followed by a decrease in the real interest rate that will depress savings and/or capital accumulation. In other words, the initial wage increase will be offset by a decrease of the real interest rate. In the case where government expenditure is financed through labor income taxes, there is one force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing labor income tax rates makes smaller the share of consumption out of wage income in the first period.
of life, thus making sunspots more likely to occur. Different from Cazzavillan and Pintus (2006), increasing labor income tax rates can not change the sensitivity of the interest rate with respect to variations in the capital stock. Considering these two reasons, it is expected that the larger labor income tax rates, the higher the impact of the wage variation on savings (that is, the lower the consumption–to–wage ratio) that is required for local indeterminacy to occur. However, in the case where government expenditure is financed through capital income taxes, there is one force which tends to dampen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make larger the lower bound of the ratio (between savings and wage income) for indeterminacy, thus making sunspots less likely to occur. At the same time, there is another force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make the after-tax interest rate more and more negatively sensitive to variations in the capital stock, thus making sunspots more likely to occur. When the former effect dominates the latter effect, increasing capital income tax rates will make local indeterminacy less likely to arise. Our numerical results indicate that in such a framework, even with income taxes, local indeterminacy does not seem plausible because it requires that both the ratio of total consumption over output and the elasticity of input substitution take too small values, which are not in accordance with empirical evidence.

The paper is organized as follows. In Section 2, we consider the case where government expenditure is financed through labor income taxes, establish the existence of a normalized steady state and present the geometrical method used for the local dynamic analysis and our main results on local indeterminacy. In Section 3, we consider the case where government expenditure is financed through capital income taxes, establish the existence of a normalized steady state and present the geometrical method used for the local dynamic analysis and our main results on local indeterminacy. Section 4 gathers some concluding comments. All the proofs are gathered in a final appendix.
2. Model 1: Labor income tax finance

As in Cazzavillan and Pintus (2004), we consider a competitive, non-monetary, overlapping generations model with production. The model involves a unique perishable good, which can be either consumed or saved as investment. Identical competitive firms all face the same technology. Identical households live for two periods. The agent consumes in both periods, supplies labor and saves when young. When old, her saved income is rented as physical capital to the firm.

Assuming additively separable preferences, the household born at time \( t \geq 0 \) maximizes her lifetime utility

$$\max_{c_{1t}, \lambda_t, c_{2t+1}} \left[ U_1(c_{1t}/B) - U_3(\lambda_t) + \beta U_2(c_{2t+1}) \right]$$

subject to the constraints

$$c_{1t} + z_t = (1 - \tau_{w_t}) \Omega_t \lambda_t \quad (1)$$

$$c_{2t+1} = R_{t+1} z_t \quad (2)$$

$$c_{1t} \geq 0, \ c_{2t+1} \geq 0, \ \lambda \geq \lambda_t \geq 0, \text{ for all } t \geq 0,$$

where \( \lambda_t, c_{1t} \) and \( z_t \) are labor, consumption and saving, respectively, of the individual of the young generation, \( c_{2t+1} \) is the consumption of the same individual when old, and \( \Omega_t > 0 \) and \( R_{t+1} > 0 \) are the real wage at time \( t \) and the gross interest rate at time \( t + 1 \). Moreover, \( \tau_{w_t}, \beta \in (0, 1), B > 0 \) and \( \lambda \) are the labor income tax rate, the discount factor, a scaling parameter and the maximum amount of labor supply, respectively.

The preferences satisfy the following condition as in Cazzavillan and Pintus (2004).

**Assumption 1.** The functions \( U_1(c/B), U_3(\lambda) \) and \( U_2(c) \) are defined and continuous on the set \( R_+ \). In addition, they are \( C^r \), for \( r \) large enough, on the set \( R_{++} \), with \( U_1'(c/B) > 0, \)
\( U'_2(c) > 0, U'_3(\lambda) > 0, U''_1(c/B) < 0, U''_2(c) < 0, U''_3(\lambda) > 0. \lim_{\lambda \to \overline{\lambda}} U'_3(\lambda) = +\infty, \) where \( \overline{\lambda} > 1, \) and \( \lim_{\lambda \to 0} U'_3(\lambda) = 0. \) Moreover, \( 0 < R_1(c/B) \equiv -(c/B)U''_1(c/B)/U'_1(c/B) < 1, 0 < R_2(c) \equiv -cU''_2(c)/U'_2(c) < 1, \) and \( R_3(\lambda) \equiv \lambda U''_3(\lambda)/U'_3(\lambda) > 0. \)

The conditions \( 0 < R_1(c/B) < 1 \) and \( 0 < R_2(c) < 1 \) are used to ensure that consumption and leisure are gross substitutes, and that the saving function is increasing with \( R. \) For example, we can assume that \( U_1(c/B) = \frac{(c/B)^{1-\alpha_1}}{1-\alpha_1}, \) \( U_2(c_2) = \frac{(c_2)^{1-\alpha_2}}{1-\alpha_2} \) and \( U_3(\lambda) = \lambda^{1+\alpha_3} \frac{1}{1+\alpha_3}. \) When \( 0 < \alpha_1 < 1, 0 < \alpha_2 < 1 \) and \( 0 < \alpha_3, \) these equations satisfy assumption 1.

When the solution of the maximization problem is interior, the first order conditions are given by

\[
U'_1(c_{1t}/B)/B = \beta R_{t+1} U'_2(c_{2t+1}) = U'_3(\lambda_t) [ (1 - \tau_{wt}) \Omega_t ]. \tag{3}
\]

Using the first order conditions, the current consumption can be written as follows

\[
c_{1t} = B \left( U'_1 \right)^{-1} \left( \frac{BU'_3(\lambda_t)}{(1 - \tau_{wt}) \Omega_t} \right), \tag{4}
\]

and the savings of the young agent born at time \( t \) are\(^2\)

\[
z_t = (1 - \tau_{wt}) \Omega_t \lambda_t - B \left( U'_1 \right)^{-1} \left( \frac{BU'_3(\lambda_t)}{(1 - \tau_{wt}) \Omega_t} \right). \tag{5}
\]

Multiplying both terms of the last equality in Eq. (3) by \( z_t \) yields

\[
\beta U'_2(c_{2t+1}) c_{2t+1} = \frac{z_t U'_3(\lambda_t)}{(1 - \tau_{wt}) \Omega_t}, \text{ or, } R_{t+1} z_t = u_2^{-1}\left( \frac{z_t U'_3(\lambda_t)}{\beta (1 - \tau_{wt}) \Omega_t} \right), \tag{6}
\]

where \( u_2(c_{2t+1}) = U'_2(c_{2t+1}) c_{2t+1} \) is a strictly increasing function of \( c_{2t+1}. \)

\(^2U'_1(c_{1t}/B) \) is decreasing and invertible in view of assumption 1.
The perishable output \((y)\) is produced using capital \((k)\) and labor \((\lambda)\),

\[
y = AF(k, \lambda) = A\lambda f(a),
\]

where \(a = \frac{k}{\lambda}\) and \(A > 0\) is a scaling factor. The competitive factor market implies that the real wage rate and the real gross rate of return on capital stock are

\[
\Omega(a) \equiv A \left[ f(a) - af'(a) \right] = A\omega(a),\
R(a) \equiv Af'(a) + 1 - \delta = A\rho(a) + 1 - \delta,
\]

where \(0 \leq \delta \leq 1\) is the constant depreciation rate of capital.\(^3\) If we consider the CES production function, the reduced production function can be given by

\[
f(a) = A(sa^{-\eta} + 1 - s)^{-\frac{1}{\eta}} \text{ if } \eta \neq 0,
\]

\[
= Aa^s \text{ if } \eta = 0,
\]

where \(\eta > -1\) determines the elasticity of input substitution through \(\sigma = 1/(1+\eta)\), while \(0 < s < 1\) governs the share of capital income in production.

As in Schmitt-Grohe and Uribe (1997) and Gokan (2006), at each point in time, the government finances its **constant** expenditure through labor income taxes, i.e.,

\[
g = \tau\omega \Omega(a_t)\lambda_t > 0.
\]

\(^3\)The reduced production function \(y/\lambda = Af(a)\) is a continuous function of the capital-labor ratio \(a = k/\lambda \geq 0\) and has continuous derivatives of all required orders for \(a > 0\), with \(f'(a) > 0\), \(f''(a) < 0\). In particular, the marginal productivity of capital \(A\rho(a) = Af'(a)\) is a decreasing function of \(a\), while the marginal productivity of labor \(A\omega(a) = A[f(a) - af'(a)]\) is increasing with \(a\).
Using the fact that at the equilibrium \( k_{t+1} = z_t \) holds, we can easily derive the dynamic system characterizing equilibrium paths of \((k_t, a_t)\).

\[
k_{t+1} = \Omega(a_t)\frac{k_t}{a_t} - B(U'_3)\left(\frac{BU'_3(k_t^a)}{\Omega(a_t)k_t^a} - g\right), \quad (9)
\]

\[
R(a_t + 1)k_{t+1} = u_2^{-1}\left\{\frac{k_{t+1}(k_t^a)U_3(k_t^a)}{\beta[\Omega(a_t)k_t^a - g]}\right\}. \quad (10)
\]

### 2.1. Steady state existence

A steady state is a pair \((k^*, a^*)\) such that.

\[
k^* = \omega(a^*) \frac{k^*}{a^*} - B(U'_3)\left(\frac{BU'_3(k^*a^*)}{\omega(a^*)k^* - ga^*} - g\right),
\]

\[
\omega(a^*) + 1 - \delta = \frac{1}{k^*} u_2^{-1}\left\{\frac{k^* U'_3(k^*)}{\beta[\omega(a^*)k^* - ga^*]}\right\}. \quad (11)
\]

To simplify the algebra, we follow the procedure used in Cazzavillan and Pintus (2004) and use the parameters \(A\) and \(B\) to normalize the steady state.

**Proposition 1.** Under those assumptions on the utility and production functions, \((k^*, a^*) = (1, 1)\) is a normalized steady state (NSS) of the dynamic system (9) and (10) if and only if \(g\) is not too large, \(A^*\omega(1) + 1 > g + 1, \beta u_2[\omega(1)(1 + g) + 1 - \delta] < U'_3(1)\) and \(\lim_{c \to 0} c U'_3(c) < \frac{A^*\omega(1) - 1 - \delta}{\beta[A^*\omega(1) - g]} U'_3(1)\), where \(A^*\) is the unique solution of \(A\rho(1) + 1 - \delta = u_2^{-1}\left\{\frac{U'_3(1)}{\beta[A^*\omega(1) - g]}\right\} \).

**Proof.** See Appendix A.1. ■

Multiplicity of steady states may arise in our model. For brevity, we just analyze the local
dynamics around the NSS.\(^4\)

2.2. Local dynamics analysis

Let us linearize the dynamic system (9) and (10) around the NSS (1, 1). We shall define \(\varepsilon_\Omega\) and \(\varepsilon_R\) as the elasticities of the functions \(\Omega(a)\) and \(R(a)\) both evaluated at the NSS. Moreover, let 
\[
\theta \equiv \frac{\Omega(a^*)}{a^*} = \Omega(1) = A^* \omega (1) > g + 1, \quad R_1 \equiv R_1(\frac{\omega}{A}), \quad R_2 \equiv R_2(c^*_2), \quad \text{and} \quad R_3 \equiv R_3(1).
\]
Then, we have the following proposition.

**Proposition 2.** The two-dimensional system (9) and (10) defines uniquely a local dynamics near the NSS \((k^*, a^*) = (1, 1)\). The linearized dynamics for the deviations \(dk = k - k^*, \ da = a - a^*\) are determined by the determinant \(D\) and the trace \(T\) of the Jacobian matrix associated with Eqs. (9) and (10).

\[
dk_{t+1} = [\theta + \frac{\theta - 1 - g}{R_1}(R_3 - \frac{g}{\theta - g})]dk_t + [\theta(\varepsilon_\Omega - 1) - \frac{\theta - 1 - g}{R_1}(R_3 + \frac{\theta \varepsilon_\Omega - g}{\theta - g})]da_t, \quad (12)
\]

\[
|\varepsilon_R|da_{t+1} = -\left\{ \frac{R_3}{1 - R_2} - \frac{g}{(\theta - g)(1 - R_2)} + \frac{R_2}{1 - R_2}[\theta + \frac{\theta - 1 - g}{R_1}(R_3 - \frac{g}{\theta - g})]\right\}dk_t +
\]

\[
\left\{ \frac{R_3}{1 - R_2} + \frac{\theta \varepsilon_\Omega - g}{(\theta - g)(1 - R_2)} - \frac{R_2}{1 - R_2}[\theta(\varepsilon_\Omega - 1) - \frac{\theta - 1 - g}{R_1}(R_3 + \frac{\theta \varepsilon_\Omega - g}{\theta - g})]\right\}da_t \quad (13)
\]

where \(g = \tau_{w}^{NSS}\Omega(1) = \tau_{w}^{NSS}\theta\) and \(\tau_{w}^{NSS} \in (0, 1)\) is the steady state labor income tax rate. Moreover, the expressions of \(D\) and \(T\) are given by

\[
T = \frac{1}{|\varepsilon_R|(1 - R_2)} \left\{ R_3 + \frac{\theta \varepsilon_\Omega - g}{\theta - g} - R_2 \left[ \theta(\varepsilon_\Omega - 1) - \frac{\theta - 1 - g}{R_1} \left( R_3 + \frac{\theta \varepsilon_\Omega - g}{\theta - g} \right) \right] \right\} + \quad (14)
\]

\[
\theta + \frac{\theta - 1 - g}{R_1}(R_3 - \frac{g}{\theta - g}),
\]

\(^4\)Thanks to Yoichi Gokan for pointing this out to us. By selecting appropriately \(A\) and \(B\) and imposing some limiting conditions, we can normalize one steady state at (1, 1).
\[D = \frac{\theta \varepsilon_{\Omega} (1 + R_3)}{|\varepsilon_R| (1 - R_2)}.\] (15)

A simple way to analyze the local dynamics of the normalized steady state is to observe the variation of the trace \(T\) and the determinant \(D\) in the \((T, D)\) plane as some parameters are made vary continuously. In particular, we are interested in the two roots of the characteristic polynomial \(Q(\pi) = \pi^2 - \pi + D\). There is a local eigenvalue which is equal to \(+1\) when \(1 - T + D = 0\). It is represented by the line (AC) in Fig. 1. Moreover, one eigenvalue is \(-1\) when \(1 + T + D = 0\). That is to say, in this case, \((T, D)\) lies on the line (AB). Finally, the two roots are complex conjugate of modulus 1, whenever \((T, D)\) belongs to the segment \([BC]\) which is defined by \(D = 1, |T| \leq 2\). Since both roots are zero when both \(T\) and \(D\) are 0, then, by continuity, they have both a modulus less than one iff \((T, D)\) lies in the interior of the triangle ABC, which is defined by \(|T| < |1 + D|, |D| < 1\). The steady state is then locally indeterminate given that there is a unique predeterminate variable \(k\). If \(|T| > |1 + D|\), the stationary state is a saddle-point. Finally, in the complementary region \(|T| < |1 + D|, |D| > 1\), the steady state is a source.

The diagram below can also be used to study local bifurcations. When the point \((T, D)\) crosses the interior of the segment \([BC]\), a Hopf bifurcation is expected to occur. If, instead, the point crosses the line (AB), one root goes through \(-1\). In that case, a flip bifurcation is expected to occur. Finally, when the point crosses the line (AC), one root goes through \(+1\), one expects an exchange of stability between the NSS and another steady state through a transcritical bifurcation.

As in Cazzavillan and Pintus (2004), we focus on two parameters, the elasticity of capital–labor substitution (\(\sigma\)) and the relative curvature of the second-period utility function \(R_2\). To be more precise, we will fix the technology, i.e. \(\theta\), the elasticities \(\varepsilon_{\Omega}\) and \(\varepsilon_R\), as well as \(R_1\) and \(R_3\), and make \(R_2\) vary continuously in the interval \((0, 1)\). This means that we will consider the parametrized curve \((T(R_2), D(R_2))\) when \(R_2\) lies in the interval \((0, 1)\). From the expressions of \(D\) and \(T\) given
in proposition 2, we find that \((T(R_2), D(R_2))\) describes a half-line \(\Delta\) which starts from the point \((T_0(\sigma), D_0(\sigma))\) for \(R_2 = 0\), where \(T_0(\sigma)\) and \(D_0(\sigma)\) are the trace in (14) and the determinant in (15) evaluated at \(R_2 = 0\). In addition, the slope of \(\Delta\) is

\[
\frac{D'(R_2)}{T'(R_2)} = \frac{\theta \varepsilon_\Omega (1 + R_3)}{\theta - g} (\theta - 1 - g) \left( \frac{1}{R_1} - 1 \right) + \theta + R_3 (1 + \frac{\theta - 1 - g}{R_1}) - \frac{g}{\theta - g} \left( 1 + \frac{\theta - 1 - g}{R_1} \right)
\]

and does not depend on \(R_2\).

Using the same method as in Cazzavillan and Pintus (2004, p. 464), we express the elasticities \(\varepsilon_\Omega\) and \(\varepsilon_R\) as functions of the depreciation rate \(\delta\), the share of capital in total income \(0 < s(a) = a \rho(a)/f(a) < 1\), and the elasticity of capital–labor substitution \(\sigma(a) \geq 0\). It is easy to find that

\[
\varepsilon_\Omega = \frac{s(a)}{\sigma(a)} \text{ and } |\varepsilon_R| = \mu(a) \frac{1 - s(a)}{\sigma(a)},
\]

where \(\mu(a) = \frac{s(a) \theta(a)}{s(a) \theta(a) + (1-s(a))(1-\delta)} \in (0,1]\) (see Cazzavillan and Pintus 2004, footnote 3 on p. 464).

Moreover, the coordinates of the origin of the half-line \(\Delta(\sigma)\) as functions of the elasticity parameter \(\sigma\) are:

\[
T_0(\sigma) = \theta + \frac{1 - g}{R_1} \left( R_3 - \frac{g}{\theta - g} \right) + \frac{\sigma \left( R_3 - \frac{g}{\theta - g} \right) + \frac{s\theta}{\theta - g}}{\mu(1-s)},
\]

\[
D_0(\sigma) = \frac{s\theta(1 + R_3)}{\mu(1-s)} \geq 0,
\]

where \(s = s(a^*), \theta = \theta(a^*), \mu = \mu(a^*) = \frac{s(a^*) \theta(a^*)}{s(a^*) \theta(a^*) + (1-s(a^*))(1-\delta)},\) and \(\sigma = \sigma(a^*)\). We can easily see that the slope of the half-line \(\Delta(\sigma)\) is

\[
\frac{\theta(1 + R_3)}{\frac{1}{1+\frac{\theta - 1 - g}{R_1}} R_3 - \frac{g}{\theta - g}} + \frac{s\theta(1 + R_3)}{\frac{1}{1+\frac{\theta - 1 - g}{R_1}} (1 - s) + \frac{\sigma (\theta - 1 - g)}{\theta - g} \frac{1}{\frac{1}{1+\frac{\theta - 1 - g}{R_1}}} - \frac{\theta}{R_1}}.
\]

**Assumption 2.** \(R_3 > \frac{g}{\theta - g} - \frac{\theta}{1+\frac{\theta - 1 - g}{R_1}} = \tau_{wSS}^{NSS} - \frac{\theta}{1+\frac{\theta - 1 - g}{R_1}}\). It corresponds to the case of small distortionary labor income tax rates, that is, \(\tau_{wSS}^{NSS}\) not large. This condition can be met for a
sufficiently high $R_3$ (if labor supply elasticity is finite), so that the slope of $\Delta(\sigma)$ is positive.\textsuperscript{5}

To understand the main results, it is useful to relate the parameters $\theta$ and $\tau_{w}^{NSS}$ to the consumption–to–wage ratio. It is easy to show that $c_1/(\Omega \lambda) = \frac{\theta(1-\tau_{w}^{NSS})-1}{\theta}$ (or, $c_1/[(1 - \tau_{w}^{NSS})\Omega \lambda] = \frac{\theta-\gamma-1}{(\theta-\gamma)}$). From this equation, one can recover the results by Cazzavillan and Pintus (2004) by setting $\tau_{w}^{NSS} = 0$.

If $s$ and $\theta$ are kept fixed and $\sigma$ is regarded as an independent parameter, we find that as $\sigma$ increases from zero to $+\infty$, the point $(T_0(\sigma), D_0(\sigma))$ moves along a flat half-line $\Delta_1$. More precisely, $T_0(\sigma)$ increases from a finite number to $+\infty$ along the flat line ($\Delta_1$), but $D_0(\sigma)$ doesn’t change. In addition, $\Delta(\sigma)$ pivots rightward and it has a positive slope when $\sigma = 0$, and it is horizontal when $\sigma = +\infty$, but the origin $(T_0(\sigma), D_0(\sigma))$ moves to the right along the line $\Delta_1$, when $\sigma$ varies from zero to $+\infty$.

In order to get local indeterminacy, first, $D_0(\sigma)$ should be less than 1, which requires that $s$ and $\theta$ be small enough, i.e., a sufficiently low share of capital in total income and a sufficiently low ratio of consumption while young to saving ($c_1^* = \theta(1 - \tau_{w}^{NSS}) - 1$ in the NSS).\textsuperscript{6} As Cazzavillan and Pintus (2004) point out, the latter requirement is crucial to local indeterminacy. Adding endogenous labor income tax rates will be helpful to local indeterminacy since in our case, the share of first period consumption over the wage income $c_1/(\Omega \lambda) = \frac{\theta(1-\tau_{w}^{NSS})-1}{\theta}$ is smaller than that in Cazzavillan and Pintus (2004). Second, we should impose some other restrictions as in Cazzavillan and Pintus (2004, the first and second paragraphs on p. 466).\textsuperscript{7}

\textsuperscript{5}In Lloyd-Braga et al. (2007), they assume that capital externalities are almost zero, $s \leq 1/2$, $\gamma \geq 1$ and $\alpha \geq \alpha_1$ (see, assumption 5 in their paper) to ensure that the slope of $\Delta$ is positive.

\textsuperscript{6}In Cazzavillan and Pintus (2004), they show that the relative curvature of the disutility of labor ($R_3$) has to be small in order to make $D_0(\sigma)$ less than 1. In Cazzavillan and Pintus (2006), $R_3$ has to be sufficiently high in order to make the slope of $\Delta_1$ positive and small enough. In our model, $R_3$ has to be sufficiently high in order to make the slope of $\Delta(\sigma)$ positive.

\textsuperscript{7}In other words, local indeterminacy requires complementary inputs ($\sigma$ is not large) and $R_1 \approx 1$ (the relative curvature of the first period consumption is close to the logarithmic specification). $R_2$ is not too close to 1, since local indeterminacy requires the generic point $(T(R_2), D(R_2))$ to lie in the interior of the stability triangle ABC, provided that $\Delta(\sigma)$ intersects the triangle ABC.
Following Cazzavillan and Pintus (2004), we consider the case (I) where $D_0(\sigma) < 1$, $T_0(0) < 1 + D_0(\sigma)$, $\text{slope}_\Delta(\sigma) > \text{slope}_\Delta(\bar{\sigma})$ and the latter slope ($\text{slope}_\Delta(\bar{\sigma})$) is bigger than 1. Here $\bar{\sigma}$ is the value of $\sigma$ such that the line $\Delta_1$ intersects the line $(AC)$. It is easy to know that the half-line $\Delta(\sigma)$ intersects the interior of the segment $BC$ for $\sigma$ in $(0, \sigma_H)$, where $\sigma_H$ is the value of $\sigma$ such that $\Delta(\sigma)$ goes through $C$. Then we know that, for all $\sigma$ in $(\bar{\sigma}, \sigma_H)$, the half-line $\Delta(\sigma)$ intersects not only the line $(AC)$ at $R_2 = R_{2T}$, but also the segment $BC$ at $R_2 = R_{2H}$. When $\sigma$ moves beyond $\sigma_H$, $\Delta(\sigma)$ will not cross the interior of the segment $BC$, but it can cross the line $AC$ up to $\sigma = \sigma_T$, where $\sigma_T$ is the value of $\sigma$ such that the $\text{slope}_\Delta(\sigma)$ is one. When $\sigma > \sigma_T$, the $\text{slope}_\Delta(\sigma)$ is less than one. We provide these parameters here.\(^8\)

\[
R_{2H} = 1 - (1 + R_3) \theta \chi (\theta) > 0, \text{ where } \chi (\theta) = \frac{s\theta + (1 - s)(1 - \delta)}{(1 - s)\theta}.
\]

\[
\sigma_H = \frac{s}{\chi(\theta)} \left( \frac{(1 - R_{2H})}{R_1} \chi_1 - \frac{\theta \chi(\theta)}{\theta - g} - \theta \chi(\theta) \frac{R_{2H} g - \theta - g}{\theta - g} + \theta \chi(\theta) \frac{R_2 H R_1}{\chi(\theta)} \right),
\]

where $\chi_1 = 2 - \theta - \frac{\theta - 1}{R_1} \left( R_3 - \frac{g}{\theta - g} \right)$.

\[
\sigma_T = \frac{s \theta \left[ 1 + R_3 - \left( 1 - \frac{1}{\theta - g} \right) \left( \frac{1}{R_1} - 1 \right) \right]}{\theta + \left( R_3 - \frac{g}{\theta - g} \right) \left( 1 - \frac{\theta - 1}{R_1} \right)}.
\]

\[
\bar{\sigma} = \frac{s (\theta - 1 - g) \left( \frac{\theta \chi(\theta)}{\theta - g} - \frac{F_1}{R_1} \right) + s \theta \chi(\theta) R_3 + s (1 - \theta)}{F_1 \chi(\theta)}, \text{ where } F_1 = R_3 - \frac{g}{\theta - g}.
\]

\[
R_{2T} = \frac{\chi_2 + \frac{\sigma \chi(\theta) F_1}{\theta - \chi(\theta) R_3 - \theta \chi(\theta) \frac{\theta - 1}{\theta - g}}}{\chi_2 + \frac{\chi(\theta)}{s} \left[ \theta (s - \sigma) - \frac{\theta - 1}{R_1} \left( \frac{s}{\theta - g} + \sigma F_1 \right) \right]} , \text{ where } \chi_2 = \theta - 1 + \frac{F_1 (\theta - 1 - g)}{R_1}.
\]

In fact, four possible dynamic regimes in the case (I) are the same as in Cazzavillan and Pintus (2004, Fig. 1 on pp. 463, 466) except that the critical values of the independent parameter $\sigma$ and the bifurcation parameter $R_2$ are different from those in their original model. We summarize these

\[^8\text{For how to derive these parameters, see the appendix A.2. in Cazzavillan and Pintus (2004). It means that } \sigma_H \text{ is the solution of } T(R_{2H}) = 2; \sigma_T \text{ is the solution of } \text{slope}_\Delta(\sigma) = 1; R_{2H} \text{ is the solution of } D(R_2) = 1; R_{2T} \text{ solves } T(R_2) = 1 + D(R_2).\]
results in the following four cases.

**Case 1.** $0 < \sigma < \varphi$. The point $(T_0(\sigma), D_0(\sigma))$ lies inside the triangle ABC, and the slope satisfies the condition $\text{slope}_\Delta(\sigma) > \text{slope}_\Delta(\varphi) > 1$. The NSS is locally indeterminate for $0 < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$.

**Case 2.** $\varphi < \sigma < \sigma_H$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC but the half line $\Delta(\sigma)$ crosses both (AC) and the interior of the segment BC. The NSS is a saddle-point for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation and exchanges stability with another steady state at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$.

**Case 3.** $\sigma_H < \sigma < \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $\text{slope}_\Delta(\sigma) > 1$, i.e. the half-line $\Delta(\sigma)$ crosses the line (AC). The NSS is a saddle for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$.

**Case 4.** $\sigma > \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $\text{slope}_\Delta(\sigma) < 1$. The NSS is a saddle for all $R_2$ in the open interval $(0, 1)$.

The following theorem summarizes the characteristics of these possible dynamic regimes as shown above.

**Theorem 1.** Let $(a^*, k^*) = (1, 1)$ be a normalized steady state which is set according to the procedure described in proposition 1. Then, under assumptions 1, 2, and those stated in the appendix A.2, the following holds.

(i) $0 < \sigma < \varphi$: the steady state $(1, 1)$ is a sink for $R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$;
(ii) $\bar{\sigma} < \sigma < \sigma_H$: the steady state $(1,1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$;

(iii) $\sigma_H < \sigma < \sigma_T$: the steady state $(1,1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$;

(iv) $\sigma > \sigma_T$: the steady state $(1,1)$ is a saddle for all $R_2$ in the open interval $(0,1)$.

**Proof.** See Appendix A.2. ■

Insert Figure 1 here.

We then turn to analyze the case (II) where the origin $(T_0(0), D_0(0))$ lies outside the triangle ABC and the slope of the half-line $\Delta(\sigma)$ is steeper than that of the line connecting the origin with the point $C$. This means that $T_0(0) > 1 + D_0(0)$, $D_0(0) < 1$, $1 < T_0(0) < 2$ and $\text{slope}_\Delta(0) > \frac{1 - D_0(0)}{2 T_0(0)}$. 9

Similar to Cazzavillan and Pintus (2004), we have the very same theorem 2 except that the critical values of the independent parameter $\sigma$ and the bifurcation parameter $R_2$ are different from those in their original model.10

**Theorem 2.** Let $(a^*, k^*) = (1,1)$ be a normalized steady state which is set according to the procedure described in proposition 1. Then, under assumptions 1, 2, and those stated in the appendix A.3, the following results hold.

---

9 The second figure can be obtained as $R_3$, given the same value of $R_3$ used to get the first figure, is smaller, because the trace is decreasing with $R_1$.

10 Three cases in figure 2 can appear.

Case 1: $0 < \sigma < \sigma_H$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC but the half line $\Delta(\sigma)$ crosses both the line (AC) and the interior of the segment BC. The NSS is a saddle-point for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation and exchanges stability with another steady state at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$.

Case 2: $\sigma_H < \sigma < \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $\text{slope}_\Delta(\sigma) > 1$, i.e. the half-line $\Delta(\sigma)$ crosses the line (AC). The NSS is a saddle for $0 < R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$.

Case 3: $\sigma > \sigma_T$. The point $(T_0(\sigma), D_0(\sigma))$ lies outside the triangle ABC and the slope satisfies the condition $\text{slope}_\Delta(\sigma) < 1$. The NSS is a saddle for all $R_2$ in the open interval $(0,1)$. 

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(i) $0 < \sigma < \sigma_H$: the steady state $(1,1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$;

(ii) $\sigma_H < \sigma < \sigma_T$: the steady state $(1,1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$;

(iii) $\sigma > \sigma_T$: the steady state $(1,1)$ is a saddle for all $R_2$ in the open interval $(0,1)$.

Insert Figure 2 here.

Perhaps the reader is interested in studying the impact of small labor income tax rates on the conditions leading to local indeterminacy, as shown in Figure 1 (or Theorem 1). The lemma 1 in the appendix shows that if $\frac{1}{1 - \tau_{w}^{NSS}} < \theta < \theta_1$ holds, indeterminacy can arise. Here $\theta_1$ is a critical value above which local indeterminacy can not arise. The next proposition will show that $\theta_1$ can be increasing in the level of labor income tax rates ($\tau_{w}^{NSS}$) provided that the tax rates ($\tau_{w}^{NSS}$) are not too large. Therefore, increasing the size of distortionary taxes from zero can enlarge the range of the values of $1/\theta$ that are compatible with indeterminacy.$^{11}$

**Proposition 3.** Under the assumptions of Theorem 1, the critical lower bound $\theta_1$ above which local indeterminacy can not arise, is increasing in the level of labor income tax rates provided that the distortionary tax rates ($\tau_{w}^{NSS}$) are not too large. Moreover, $R_3 > \frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}} - \frac{\theta}{1 + \frac{\theta - 1 - \theta \tau_{w}^{NSS}}{R_{1} w}}$ will be met if the utility function in the first period of life is close enough to logarithmic ($R_1 = 1$) and $\tau_{w}^{NSS}$ is not too large.

Insert Figure 3 here.

$^{11}$Notice that the range of the indeterminacy region is $\theta_1 - \frac{1}{1 - \tau_{w}^{NSS}}$, which is decreasing with $\tau_{w}^{NSS}$ under the assumption of $s = 0.3$ and $\delta = 1$. But the minimum level of the ratio between savings and wage (namely, $\frac{1}{\delta}$) for indeterminacy is decreasing with $\tau_{w}^{NSS}$. That is to say, the higher the steady state labor income tax rates, indeterminacy is more likely.
The following numerical example shows how the share of wage devoted to savings has to be large for local indeterminacy to arise ($\theta$ should be small) and how endogenous labor income tax rates are helpful to local indeterminacy: the latter conclusion is consistent with recent works, for example, Schmitt-Grohe and Uribe (1997) and Gokan (2006).\footnote{For a given level of $\theta$, indeterminacy is more likely, the larger the steady state labor income tax rates.} In particular, Schmitt-Grohe and Uribe (1997) have shown that, in a standard neoclassical growth model, endogenous income tax rate may itself generate indeterminacy for reasonable values of the model parameters. In addition, we illustrate, using numerical examples, our main results that increasing steady state labor income tax rates may enlarge the range of parameter values ($\sigma_H$) associated with multiple equilibria. However, in the OLG framework with consumption in both periods of life, with endogenous labor income taxes, local indeterminacy requires both complementary inputs and unrealistic values of the ratio of total consumption over output.

To fix ideas and ease comparisons with Cazzavillan and Pintus (2004), we set $s = 0.3$ and $\delta = 1$, where full capital depreciation is perfectly consistent with the time period implied by the OLG setting, and the chosen value of the capital share in total income is close to the one that Schmitt-Grohe and Uribe (1997) use. We further assume that $\tau_N^{SS}$ can take the values of 0.1, 0.12, 0.14, 0.16 and 0.18. These values can imply the two bounds of $\theta$ (i.e., $\frac{1}{1-\tau_N^{SS}}$ and $\theta_1$). The values of $R_1$ and $R_3$ must belong to the relevant intervals defined in lemma 1. And we assume that $R_1 = 0.99$ and $R_3 = 0.62$.\footnote{For how to select these proper values of $R_1$ and $R_3$, see the matlab programs which are available upon request.} Similar to Cazzavillan and Pintus (2004), we can show that total consumption, including consumption by the old agents, has to be less than 45% of output (not in agreement with averages for the OECD countries) in the case of Fig. 1.

Considering the elasticity of capital–labor substitution, we find that the condition $\sigma < \sigma_H$, which is necessary to get endogenous fluctuations, places an upper bound on $\sigma$. It is easy to find that $\sigma_H < \sigma_T$. Numerical examples show that $\sigma_T > s$ and, therefore, that $\sigma_H < s < \sigma_T$. This suggests
that $\sigma_H$ may be below the capital share, and that $\sigma_T$ may be above the capital share. In fact, we illustrate that, irrespective of the values for $R_1$ and $R_3$, $\sigma_H$ decreases when $\theta$ increases, for a given $\tau_w^{NSS}$ and; $\sigma_H$ increases when $\tau_w^{NSS}$ increases, for a given $\theta$. The former conclusion has already been found by Cazzavillan and Pintus (2004). The latter conclusion (in our model) shows that endogenous labor income tax rates are helpful to local indeterminacy.

The former conclusion has already been found by Cazzavillan and Pintus (2004). The latter conclusion (in our model) shows that endogenous labor income tax rates are helpful to local indeterminacy.

Insert Table 1 here.

We are now in a position to intuitively explain why endogenous labor income tax rates are helpful to local indeterminacy. Cazzavillan and Pintus (2004) have already shown that when intertemporal substitution in consumption across periods is introduced, endogenous fluctuations require very low values of the propensity to consume out of wage income of the young generation (in our model, $(1 - \tau_w^{NSS}) - \frac{1}{\theta}$). In addition, endogenous fluctuations require elasticities of capital–labor substitution that are well below the share of capital in total income. We find that (1) for a given technology $\theta$, adding labor income tax rates ($\tau_w^{NSS}$) will make the ratio of consumption while young to saving smaller, thus making sunspots more likely to occur and; (2) for a given technology $\theta$, adding labor income tax rates will make the bound on $\sigma$ associated with multiple equilibria ($\sigma_H$) larger although this bound is still less than the share of capital in total income. To be more precise, we provide the following intuitive interpretation. Endogenous fluctuations arise due to the interaction of two conflicting effects: when the capital stock increases, it leads to an increase in wage rate and, therefore, an increase in savings which leads the capital stock in the next period to be higher. However, capital accumulation is followed by a decrease in the real interest rate that will depress savings and/or capital accumulation. In other words, the initial wage increase will be offset by a decrease of the real interest rate. In our model, there is one force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing labor income tax rates makes smaller the share

\footnote{Again, $R_1$ and $R_3$ must belong to the relevant intervals defined in lemma 1.}
of consumption out of wage income in the first period of life, thus making sunspots more likely to occur.\textsuperscript{15} Different from Cazzavillan and Pintus (2006), increasing tax rates can \textit{not} change the sensitivity of the interest rate with respect to variations in the capital stock (the elasticity of $R$ with respect to $k$ is $\varepsilon_{R,k} = \mu(a) \frac{s(a) - 1}{\sigma(a)} < 0$ and does \textit{not} depend on $\tau_{w}^{NSS}$). Considering these two reasons, it is expected that the larger labor income tax rates, the higher the impact of the wage variation on savings (that is, the lower the consumption–to–wage ratio) that is required for local indeterminacy to occur.

Lastly, we need point out that the conflicting effects on savings through wage rate and interest rate work neither in the infinite-horizon model as in Schmitt-Grohe and Uribe (1997), nor in the finance constrained economy as in Gokan (2006). Now it is well known that labor income tax rates only are enough to generate indeterminacy in the Ramsey model as long as constant government expenditure is financed by labor income taxes, but this property does \textit{not} hold in the OLG model with consumption in both periods of life.

\textbf{3. Model 2: Capital income tax finance}

In this section, we introduce constant government expenditure financed by capital income taxes in the OLG model studied in Cazzavillan and Pintus (2004). The household budget constraints are described by the following new equations:

\begin{align*}
    c_{1t} + z_{t} &= \Omega_{t} \lambda_{t}, \\
    c_{2t+1} &= \bar{R}_{t+1} z_{t},
\end{align*}

\textsuperscript{15}Although the ratio between savings and wage income is fixed $(1/\theta)$ by choosing the parameters $A$ and $B$, we find that the lower bound of this ratio for indeterminacy $(1/\theta_{1})$ is decreasing with the steady state labor income tax rate. In other words, increasing labor income tax rates can make smaller the lower bound of the ratio (between savings and wage income) for indeterminacy, thus making sunspots more likely to occur.
where \( \bar{R}_{t+1} > 0 \) is the after–tax gross interest rate at time \( t + 1 \).

When the solution of the maximization problem is interior, the first order conditions are given by

\[
U'_1(c_{1t}/B)/B = \beta \bar{R}_{t+1} U'_2(c_{2t+1}) = U'_3(\lambda_t)/\Omega_t.
\] (20)

Using the first order conditions, the current consumption can be written as follows

\[
c_{1t} = B \left( U'_1 \right)^{-1} \left( \frac{BU'_3(\lambda_t)}{\Omega_t} \right),
\] (21)

and the savings of the young agent born at time \( t \) are

\[
\begin{align*}
z_t &= \Omega_t \lambda_t - B \left( U'_1 \right)^{-1} \left( \frac{BU'_3(\lambda_t)}{\Omega_t} \right). \\
\end{align*}
\] (22)

Multiplying both terms of the last equality in Eq. (20) by \( z_t \) yields

\[
\begin{align*}
\beta U'_2(c_{2t+1}) c_{2t+1} &= \frac{z_t U'_3(\lambda_t)}{\Omega_t}, \text{ or, } \bar{R}_{t+1} z_t = u_2^{-1} \left( \frac{z_t U'_3(\lambda_t)}{\beta \Omega_t} \right), \\
\end{align*}
\] (23)

where \( u_2(c_{2t+1}) = U'_2(c_{2t+1}) c_{2t+1} \) is an increasing function of \( c_{2t+1} \).

As in Schmitt-Grohe and Uribe (1997), at each point in time, the government can finance its constant expenditure through capital income taxes, i.e.,

\[
g = \tau_{kt} r_t k_t > 0,
\] (24)

where \( r_t \) and \( \tau_{kt} \) are the marginal productivity of capital \( (r_t = A f'(a_t)) \) and the capital income tax
rate. It is easy to show that the after-tax gross interest rate at time $t$ is

$$\tilde{R}_t = (1 - \tau_k) r_t + 1 - \delta.$$  

Using the fact that at the equilibrium $k_{t+1} = z_t$ holds, we can easily derive the dynamic system characterizing equilibrium paths of $(k_t, a_t)$.

$$R(a_{t+1})k_{t+1} = u_2^{-1}\left(\frac{k_{t+1}U_3'(k_t/a_t)}{\beta \Omega(a_t)}\right) + g, \quad (25)$$

$$k_{t+1} = \Omega(a_t)\frac{k_t}{a_t} - B(U'_1)^{-1}\left(\frac{BU_3'(k_t/a_t)}{\Omega(a_t)}\right). \quad (26)$$

3.1. Steady state existence

A steady state is a pair $(\bar{k}, \bar{a})$ such that.

$$A \rho (\bar{a}) + 1 - \delta = \frac{1}{\bar{k}} \left[u_2^{-1}\left(\bar{k}U'_1(\bar{k}/\bar{a})\right) + g\right], \quad (27)$$

$$\bar{k} = A \omega (\bar{a}) \frac{\bar{k}}{\bar{a}} - B(U'_1)^{-1}(\frac{BU_3'(\bar{k}/\bar{a})}{A \omega (\bar{a})}). \quad (28)$$

To simplify the algebra, we follow the procedure described in Cazzavillan and Pintus (2004) and use the parameters $A$ and $B$ to normalize the steady state.

**Proposition 4.** Under the assumptions on the utility and production functions, $(\bar{k}, \bar{a}) = (1, 1)$ is a normalized steady state (NSS) of the dynamic system (25) and (26) if and only if $g$ is not too large, $A^* \omega (1) > 1$, $\beta u_2 [\frac{\rho(1)}{\omega(1)} + 1 - \delta - g] < U'_3(1)$ and $\lim_{c \to 0} c U'_1(c) < \frac{A^* \omega (1) - 1}{A^* \omega (1)} U'_3(1)$, where $A^*$ is the unique solution of $A \rho (1) + 1 - \delta - g = u_2^{-1} \left(\frac{U'_3(1)}{\beta \omega(1)}\right)$.

**Proof.** See Appendix A.4.  ■
Multiplicity of steady states can arise in our model. For brevity, we just analyze the local dynamics around the NSS.

3.2. Local dynamics analysis

Let us linearize the dynamic system (25) and (26) around the NSS (1,1). We shall define $\varepsilon_\Omega$ and $\varepsilon_R$ as the elasticities of the functions $\Omega(a)$ and $R(a)$ evaluated at the NSS. Moreover, let $\theta \equiv \Omega(\overline{a})/\overline{a} = \Omega(1) = A^*\omega(1) > 1$, $R_1 \equiv R_1(\frac{\overline{c}}{\overline{B}^*})$, $R_2 \equiv R_2(\overline{c}_2)$, and $R_3 \equiv R_3(1)$. Then, we have the following proposition.

**Proposition 5.** The linearized dynamics generated by the two-dimensional system (25) and (26) around the NSS are determined by the determinant $D$ and the trace $T$ of the Jacobian matrix associated with Eqs. (25) and (26).

\[
dk_{t+1} = \left[ \theta + \frac{\theta - 1}{R_1} R_3 \right] dk_t + \left[ \theta (\varepsilon_\Omega - 1) - \frac{\theta - 1}{R_1} (R_3 + \varepsilon_\Omega) \right] da_t,
\]  

(29)

\[
da_{t+1} = -\frac{1}{|\varepsilon_R|} \{ 1 - \tau_k^{\text{ss}} \left[ \theta (\varepsilon_\Omega - 1) - (R_3 + \varepsilon_\Omega) \left( \frac{\theta - 1}{R_1} R_3 + 1 \right) \right] - \left[ \theta (\varepsilon_\Omega - 1) - \frac{(\theta - 1)(R_3 + \varepsilon_\Omega)}{R_3} \right] \} da_t,
\]  

(30)

where $\mu \equiv \frac{A^{\ast} \rho(1)}{A^{\ast} \rho(1) + 1 - \delta} \in (0, 1)$ and $\tau_k^{\text{ss}} \in (0, 1)$ is the steady state capital income tax rate around the NSS. Moreover, the expressions of $D$ and $T$ are given by

\[
T = \frac{1}{|\varepsilon_R|} \left[ \theta (\varepsilon_\Omega - 1) - \frac{\theta - 1}{R_1} (R_3 + \varepsilon_\Omega) \right] + \theta + \frac{\theta - 1}{R_1} R_3
\]  

(31)

\[
- \frac{1}{|\varepsilon_R| (1 - R_2)} \left[ \theta (\varepsilon_\Omega - 1) - (R_3 + \varepsilon_\Omega) \left( \frac{\theta - 1}{R_1} + 1 \right) \right].
\]
\[ D = \frac{\theta \varepsilon \Omega (1 + R_3)}{|\varepsilon_R| (1 - R_2)} (1 - \tau_k^{nss} \mu) > 0. \] (32)

In this model, the slope of \( \Delta \) is

\[ \frac{D'(R_2)}{T'(R_2)} = \frac{\theta \varepsilon \Omega (1 + R_3)}{R_3 \left( \frac{\theta - 1}{R_1} + 1 \right) + \varepsilon \Omega \left( \theta - 1 \right) \left( \frac{1}{R_1} - 1 \right) + \theta} > 0. \] (33)

and does not depend on \( R_2 \). Moreover, the coordinates of the origin of the half-line \( \Delta(\sigma) \) as functions of the elasticity parameter \( \sigma \) are:

\[
T_0(\sigma) = \frac{\sigma R_3 + s}{\mu(1 - s)} + \theta + \left( \frac{\sigma - 1}{\mu(1 - s)} \right) R_3 \left( \frac{1}{R_1} + 1 \right) + s(\theta - 1)(1 - R_1) + \theta, \\
D_0(\sigma) = \frac{s(\theta + 1 - R_3)}{\mu(1 - s)} (1 - \tau_k^{nss} \mu) \geq 0,
\]

where \( s = s(\overline{\sigma}), \theta = \theta(\overline{\sigma}), \mu = \mu(\overline{\sigma}) = \frac{s(\overline{\sigma})}{s(\overline{\sigma}) + (1 - s(\overline{\sigma}))(1 - \sigma)}, \) and \( \sigma = \sigma(\overline{\sigma}) \). Therefore, the slope of the half-line \( \Delta(\sigma) \) can be written as follows

\[
\frac{s(\theta + 1 - R_3)R_3}{s(\sigma - 1)(1 - R_1) + s(\sigma - 1)(1 - R_1) + \theta R_1}. \]

**Assumption 3.** \( R_3 > \frac{\tau_k^{nss} \frac{\varphi - 1}{\varphi + 1 - \sigma}}{(\varphi + 1 - \sigma)R_1} \left[ R_3 (\theta - 1 + R_1) + \theta R_1 \right] \). It corresponds to the case of small distortionary capital income tax rates, that is, \( \tau_k^{nss} \) not large. This condition can be met for a sufficiently high \( R_3 \) (if labor supply elasticity is finite), so that \( T_0(\sigma) \) is an increasing function of \( \sigma \).

In this case, \( T_0(\sigma) \) increases from \( T_0(0) \) to \( +\infty \) along the half line \( \Delta_1 \), as \( \sigma \) increases from zero to \( +\infty \).\(^{16}\)

To understand the main results, it is useful to relate the parameters \( \theta \) and \( \tau_k^{nss} \) to the consumption-to-wage ratio. It is easy to show that \( c_1/(\Omega \lambda) = \frac{\theta - 1}{\theta} \). From this equation, one can recover the results by Cazzavillan and Pintus (2004) by setting \( \tau_k^{nss} = 0 \).

\(^{16}\) In Cazzavillan and Pintus (2004), they show that the relative curvature of the disutility of labor (\( R_3 \)) has to be small in order to make \( D_0(\sigma) \) less than 1. In Cazzavillan and Pintus (2006), \( R_3 \) has to be sufficiently high in order to make the slope of \( \Delta_1 \) positive and small enough. In our model, \( R_3 \) has to be sufficiently high in order to ensure that \( T_0(\sigma) \) is increasing with \( \sigma \).
If $s$ and $\theta$ are kept fixed and $\sigma$ is regarded as an independent parameter, we find that as $\sigma$ increases from zero to $+\infty$, the point $(T_0(\sigma), D_0(\sigma))$ moves along a flat half-line $\Delta_1$. More precisely, $D_0(\sigma)$ doesn’t change, but $T_0(\sigma)$ increases from a finite number to $+\infty$ along the flat line $\Delta_1$, when $\tau_k^{nss}$ is small. In addition, $\Delta(\sigma)$ pivots rightward and it has a positive slope when $\sigma = 0$, and it is horizontal when $\sigma = +\infty$, but the origin $(T_0(\sigma), D_0(\sigma))$ moves to the right along the line $\Delta_1$, when $\sigma$ varies from zero to $+\infty$.

As in the former model, we consider the case (I) and provide these parameters here.

\[
R_{2H} = 1 - \frac{s\theta (1 + R_3) \left(1 - \tau_k^{nss} \right)}{\mu (1 - s)},
\]

\[
\sigma_H = \frac{2 - \frac{s(1 - \tau_k^{nss})}{\mu(1-s)(1-R_{2H})} \left(\frac{\theta - 1}{R_1} + 1 - \theta\right) - \frac{s}{\mu(1-s)} \left(\theta - \frac{\theta - 1}{R_1}\right) - \theta - \frac{\theta - 1}{R_1} R_3}{\frac{1}{\mu(1-s)} \left\{ 1 - \frac{\tau_k^{nss} \mu}{1 - R_{2H}} \left[R_3 \left(\frac{\theta - 1}{R_1} + 1\right) + \theta\right] - \left(\theta + \frac{\theta - 1}{R_1} R_3\right) \right\}},
\]

\[
\sigma_T = \frac{s \left[\theta (1 + R_3) - (\theta - 1)(1/R_1 - 1)\right]}{\theta + R_3 \left(1 + \frac{1}{R_1}\right)},
\]

\[
\overline{\sigma} = \frac{\chi(\theta) \theta (1 + R_3) \left(1 - \tau_k^{nss}\right)}{\chi(\theta) R_3 \left(\frac{\theta - 1}{R_1} + 1\right) + \theta} - \frac{\tau_k^{nss} s (\theta - 1)}{1 - s} \left(\frac{1}{R_1} - 1\right) - (\theta - 1) \left(\theta + \frac{R_3}{R_1}\right) - \chi(\theta),
\]

where $\chi(\theta) = \frac{s\theta + (1-s)(1-\delta)}{(1-s)\theta}$,

\[
R_{2T} = 1 - \frac{(1 - \tau_k^{nss}) \left\{ \sigma \left[R_3 \left(\frac{\theta - 1}{R_1} + 1\right) + \theta\right] + s \left(\theta - 1\right) \right\}}{\mu (1 - s) + \left(\theta + \frac{\theta - 1}{R_1} R_3\right) \left[\sigma - \mu (1 - s)\right] - s \left(\theta - \frac{\theta - 1}{R_1}\right)}.
\]

It is easy to find that the introduction of capital income taxes can affect the critical values of these parameters. In contrast with the former results, endogenous capital income taxes are less helpful to local indeterminacy than labor income taxes. We provide the following proposition.

**Proposition 6.** The introduction of endogenous capital income taxes does not affect the critical values $\sigma_H$ and $\sigma_T$.

**Proof.** Since the slope of $\Delta(\sigma)$ does not depend on $\tau_k^{nss}$ and $slope_{\Delta(\sigma_T)} = 1$, we know that $\sigma_T$ does not depend on $\tau_k^{nss}$. From the expression of $R_{2H}$, we know that $\frac{1 - \tau_k^{nss} \mu}{1 - R_{2H}} = \frac{\mu(1-s)}{s\theta(1+R_3)} = \frac{1}{\chi(\theta) \theta(1+R_3)}$,
which does not depend on \( \tau_{nk}^{nss} \). Replacing \( \frac{1-\tau_{nk}^{nss}\mu}{1-H_2} \) with \( \frac{1}{\chi(\bar{\theta})\theta(1+R_3)} \) in the formula of \( \sigma_H \), we know that \( \sigma_H \) does not depend on \( \tau_{nk}^{nss} \). ■

Four possible dynamic regimes in the case (I) are the same as in Cazzavillan and Pintus (2004, Fig. 1 on pp. 463, 466) except that the critical values of the independent parameter \( \sigma \) and the bifurcation parameter \( R_2 \) are different from those in their model. We summarize these results in the following theorem.

**Theorem 1.** Let \( (\bar{k}, \bar{n}) = (1, 1) \) be a normalized steady state which is set according to the procedure outlined in proposition 4. Then, under assumptions 1, 3, and those stated in the appendix A.5, the following holds.

(i) \( 0 < \sigma < \bar{\sigma} \): the steady state \((1, 1)\) is a sink for \( R_2 < R_{2H} \), undergoes a Hopf bifurcation at \( R_2 = R_{2H} \), and becomes a source for \( R_2 > R_{2H} \);

(ii) \( \bar{\sigma} < \sigma < \sigma_H \): the steady state \((1, 1)\) is a saddle for \( R_2 < R_{2T} \), undergoes a transcritical bifurcation at \( R_2 = R_{2T} \), becomes a sink for \( R_{2T} < R_2 < R_{2H} \), undergoes a Hopf bifurcation at \( R_2 = R_{2H} \), and becomes a source for \( R_2 > R_{2H} \);

(iii) \( \sigma_H < \sigma < \sigma_T \): the steady state \((1, 1)\) is a saddle for \( R_2 < R_{2T} \), undergoes a transcritical bifurcation at \( R_2 = R_{2T} \), and becomes a source for \( R_2 > R_{2T} \);

(iv) \( \sigma > \sigma_T \): the steady state \((1, 1)\) is a saddle for all \( R_2 \) in the open interval \((0, 1)\).

**Proof.** See Appendix A.5. ■

For brevity, we will not turn to analyze the case (II) where the origin \((T_0(0), D_0(0))\) lies outside the triangle ABC and the slope of the half-line \( \Delta(\sigma) \) is steeper than that of the line connecting the origin with the point \( C \). This means that \( T_0(0) > 1 + D_0(0) \), \( D_0(0) < 1 \), \( 1 < T_0(0) < 2 \) and \( slope_\Delta(0) > \frac{1-D_0(0)}{2-T_0(0)} \). Similar to Cazzavillan and Pintus (2004), we may have the very same theorem 2 except that the critical values of the independent parameter \( \sigma \) and the bifurcation parameter \( R_2 \) are different from those in their original model.
Perhaps the reader is interested in studying the impact of small capital income tax rates on the conditions leading to local indeterminacy, as shown in Theorem 3. The lemma in the appendix A.5 shows that if $1 < \theta < \theta_1'$ holds, indeterminacy can arise. Here $\theta_1'$ is a critical value above which local indeterminacy can not arise. The interesting finding is that $\theta_1'$ can be decreasing in the level of capital income tax rates ($\tau_{k}^{nss}$) provided that the rates ($\tau_{k}^{nss}$) are not too large. Therefore, increasing the size of distortionary capital income taxes from zero can not enlarge the range of the values of $\theta$ that are compatible with local indeterminacy.

**Proposition 7.** Under the assumptions of Theorem 3, the critical lower bound $\theta_1'$ above which local indeterminacy can not arise is decreasing in the level of capital income tax rates provided that the distortionary tax rates ($\tau_{k}^{nss}$) are not too large. Moreover, $R_3 > \frac{\tau_{k}^{nss} \frac{s}{1-s} \theta}{(1-s)(\theta + 1-\delta)} R_1 [R_3 (\theta - 1 + R_1) + \theta R_1]$ will be met if the utility function in the first period of life is close enough to logarithmic ($R_1 = 1$) and $\tau_{k}^{nss}$ is not too large.

Insert Figure 4 here.

The following numerical example shows how the share of wage devoted to savings has to be large for local indeterminacy to arise ($\theta$ should be small) and how endogenous capital income tax rates are not helpful to local indeterminacy. Moreover, we illustrate, using numerical examples, our main results that increasing steady state capital income tax rates may make shrink the range of parameter values ($\bar{\sigma}$) associated with multiple equilibria.

To do this, we set $s = 1/3$ and $\delta = 1$, and assume that $\tau_{k}^{nss}$ can take the values of 0.1, 0.12, 0.14, 0.16, 0.18 and 0.20. These values can imply the bound of $\theta$ (i.e., $\theta_1'$). The values of $R_1$ and $R_3$ must belong to the relevant intervals defined in lemma 3. And we assume that $R_1 = 0.95$ and $R_3 = 0.82$. Considering the elasticity of capital–labor substitution, it is easy to find that $\sigma_H < \sigma_T$. Numerical examples show that $\sigma_T < s$ and, therefore, that $\sigma_H < \sigma_T < s$. This suggests that $\sigma_H$ may be below
the capital share. In fact, we illustrate that, irrespective of the values for \( R_1 \) and \( R_3 \), \( \sigma_H \) and \( \sigma_T \) do not depend on \( \tau_k^{nss} \), \( \sigma_H \) decreases when \( \theta \) increases for a given \( \tau_k^{nss} \), and \( \overline{\sigma} \) is decreasing with \( \tau_k^{nss} \) for a given \( \theta \). This conclusion shows that endogenous capital income tax rates are not helpful to local indeterminacy. Similar to Cazzavillan and Pintus (2004), we can show that total consumption, including consumption by the old agents, has to be less than 45% of output in this case.

We also find that (1) adding capital income tax rates (\( \tau_k^{nss} \)) will make larger the lower bound of the ratio (between savings and wage income, \( \frac{1}{\rho_1} \)) for indeterminacy, thus making sunspots less likely to occur and; (2) for a given technology \( \theta \), adding tax rates will make the bound on \( \sigma \) associated with multiple equilibria (\( \overline{\sigma} \)) smaller (this bound is less than the share of capital in total income). Here we provide the following intuitive interpretation. In this model, there is one force which tends to dampen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make larger the lower bound of the ratio (between savings and wage income) for indeterminacy; thus making sunspots less likely to occur. At the same time, there is another force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing capital income tax rates can make the after-tax interest rate more and more negatively sensitive to variations in the capital stock (the elasticity of \( \tilde{R} \) with respect to \( k \) is \( \varepsilon_{\tilde{R},k} = \frac{1}{1-\tau_k^{nss} \mu(a)} \frac{\mu(a)[s(a)-1]}{\sigma(a)} < 0 \) and decreases with \( \tau_k^{nss} \) for small values of \( \sigma \) when \( \sigma < \sigma_H \)), thus making sunspots more likely to occur. When the former effect dominates the latter effect, increasing capital income tax rates will be harmful to local indeterminacy.

4. Concluding Remarks

We study a version of Diamond’s OLG model modified to allow for consumption in both periods of life and endogenous income tax rates. We have shown that local indeterminacy of the steady state can
arise, when income tax rates are not too large, as long as the fraction of young-age consumption out of wage income is small enough. More importantly, we find that increasing the size of tax distortions enlarges the range of values of the consumption–to–wage ratio associated with local indeterminacy if constant government expenditure is financed through labor income taxes, while increasing the size of tax distortions can make shrink the range of values of the consumption–to–wage ratio associated with local indeterminacy if constant government expenditure is financed through capital income taxes. This result might be related to the fact that adding labor income tax rates ($r_{w}^{NSS}$) will make the ratio of consumption while young to saving smaller, thus making sunspots more likely to occur. This explains why endogenous labor income taxes are helpful to local indeterminacy in the OLG model with first-period consumption and it is also useful to understand why labor income tax rates can lead to local indeterminacy in the infinite-horizon model (Schmitt-Grohe and Uribe, 1997).

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**Appendix:**

**A.1. Proof of Proposition 1**

If $(k^*, a^*) = (1, 1)$ is a normalized steady state of the dynamic system (9) and (10), we have the following: ($c_1^*$ is the steady state of the first period consumption.)

\[
A\omega (1) - g - 1 = B(U_1')^{-1} \left( \frac{BU_3'(1)}{A\omega (1) - g} \right) = c_1^* > 0, \tag{D-1}
\]

\[
A\rho (1) + 1 - \delta = u_2^{-1} \left\{ \frac{U_3'(1)}{\beta[A\omega (1) - g]} \right\}. \tag{D-2}
\]
If $g$ is not too large, $A > \frac{g+1}{\omega(1)}$ can make $c_1^* > 0$. It is easy to find that $\beta[A\omega(1) - g]u_2[A\rho(1)+1-\delta] = U_3(1)$ and the LHS term is an increasing function of $A$. In order to obtain a unique $A^*$ satisfying $(D-2)$, we require that $\beta[A\omega(1) - g]u_2[A\rho(1)+1-\delta]A^{-\frac{g+1}{\omega(1)}} < U_3'(1)$. It is equivalent to $\beta u_2[A\rho(1)+1-\delta] < U_3'(1)$. We can easily get $B^*$ from $(D-1)$ after we pin down the unique $A^*$ from $(D-2)$. In particular, we can rewrite $(D-1)$ as follows: \( \frac{A\omega(1)-g-1}{B} U_1'(\frac{A\omega(1)-g-1}{B}) = \frac{A\omega(1)-g-1}{A\omega(1)-g} U_3'(1) \). It is easy to see that \( \frac{A\omega(1)-g-1}{B} U_1'(\frac{A\omega(1)-g-1}{B}) \) is a decreasing function of $B$. In order to have the unique $B^*$, we should impose the restriction: \( \lim_{c \to 0} c U_1'(c) < \frac{A\omega(1)-1-g}{A\omega(1)-g} U_3'(1) \).

A.2. Proof of Theorem 1

**Lemma 1.** Let $\frac{1}{1-\tau_{\text{NSS}}} < \theta < \theta_1 = \frac{\Upsilon + \sqrt{\Upsilon^2 - 4\phi}}{2\phi}$, where $\Upsilon \equiv \frac{(2-\delta)(1-s)(1-\tau_{\text{NSS}})-2s(1-\delta)-s(1-s)(1-\delta)\tau_{\text{NSS}}}{(1-s)(1-\tau_{\text{NSS}})}$ and $\phi \equiv -s(1-s)(1-\tau_{\text{NSS}})^2 + s^2 + s\tau_{\text{NSS}}(1-s) - s^2 - 2s(1-s)(1-\delta)$. Moreover, we assume that $R_1 > \bar{R}_1$ and $R_3 < R_3 < \bar{R}_3$, where $\bar{R}_3 = \frac{\theta - 1 - \theta \chi(\theta) + \frac{\chi(\theta)}{1-\tau_{\text{NSS}}} - [\theta(1-\tau_{\text{NSS}})-1] \frac{\tau_{\text{NSS}}}{1-\tau_{\text{NSS}}}}{1+\theta \chi(\theta) - \theta(1-\tau_{\text{NSS}})}$, $\bar{R}_3 = \frac{1-\theta \chi(\theta)}{\theta \chi(\theta)}$ and $R_1 = \frac{\theta(1-\tau_{\text{NSS}})-1}{\theta \chi(\theta)} \frac{R_3 - \frac{\tau_{\text{NSS}}}{1-\tau_{\text{NSS}}}}{1+\theta \chi(\theta) - \theta(1-\tau_{\text{NSS}})}$, with $\frac{s(1-s)}{(1-s)^2} = \chi(\theta)$. Then we have the following results: the origin $(T_0(0), D_0(0))$ lies inside the $ABC$ triangle and the half line $\Delta(\sigma)$ intersects the interior of the segment $BC$ at $\sigma = 0$ $(T_0(0) < 1 + D_0(0), D_0(0) < 1)$. Moreover, we have $\text{slope}_\Delta(0) > \text{slope}_\Delta(\sigma) > \text{slope}_\Delta(\sigma_H) > \text{slope}_\Delta(\sigma_T) = 1$.

**Proof.** Similar to Cazzavillan and Pintus (2004), $D_0(0) < 1$ is satisfied iff $0 < R_3 < \frac{1-\theta \chi(\theta)}{\theta \chi(\theta)}$ if $\bar{R}_3$, where $\chi(\theta) = \frac{s(1-\theta)(1-\delta)}{(1-s)\theta}$ and $\mu = \frac{s}{(1-s)\chi(\theta)}$. This requires that $\theta < \bar{\theta} \equiv \frac{\delta(1-s)}{s}$ and $s < \frac{\delta}{(1-\tau_{\text{NSS}})^{-1+\delta}} \leq \frac{1}{2}$ as $\theta > 1 + g$.

$T_0(0) < 1 + D_0(0)$ is satisfied iff $R_1 > \bar{R}_1 \equiv \frac{\theta(1-\tau_{\text{NSS}})-1}{\theta \chi(\theta)(1+R_3)} \theta - \frac{\chi(\theta)}{1-\tau_{\text{NSS}}}$ with $R_3 > \bar{R}_3 = \frac{\theta - 1 - \chi(\theta)}{\theta \chi(\theta)} \frac{1}{\tau_{\text{NSS}}}$. Since $R_1 < 1$, we need that $\bar{R}_1 < 1$, which is equivalent to

$$R_3 > \bar{R}_3 = \frac{\theta - 1 - \theta \chi(\theta) + \frac{\chi(\theta)}{1-\tau_{\text{NSS}}} - [\theta(1-\tau_{\text{NSS}})-1] \frac{\tau_{\text{NSS}}}{1-\tau_{\text{NSS}}}}{1+\theta \chi(\theta) - \theta(1-\tau_{\text{NSS}})}.$$
where \( 1 + \theta \chi(\theta) - \theta(1 - \tau_{SS}^{w}) > 0 \). When \( \theta < \bar{\theta} \equiv \frac{(2-\delta)(1-s)}{(1-\tau_{SS}^{w})(1-s)-s} \). It is easy to verify that if \( \delta > \frac{2s}{(1-\tau_{SS}^{w})(1-s)} \), the binding upper bound on \( \theta \) is \( \tilde{\theta} \), as \( \bar{\theta} < \tilde{\theta} \). Otherwise, if \( \delta < \frac{2s}{(1-\tau_{SS}^{w})(1-s)} \), the binding upper bound on \( \theta \) is \( \bar{\theta} \), as \( \bar{\theta} < \tilde{\theta} \). In addition, \( R_{3} > \bar{R}_{3} \). Then we have that \( D_{0}(0) < 1 \) and \( T_{0}(0) < 1 + D_{0}(0) \) if \( R_{1} > R_{1} \) and \( R_{3} < R_{3} < \bar{R}_{3} \), provided that either \( \theta < \bar{\theta} \), when \( \delta < \frac{2s}{(1-\tau_{SS}^{w})(1-s)} \), or \( \theta < \tilde{\theta} \), when \( \delta > \frac{2s}{(1-\tau_{SS}^{w})(1-s)} \). The inequality \( R_{3} < R_{3} < \bar{R}_{3} \) holds iff the polynomial holds.

\[
P_{1}(\theta) = \phi \theta^{2} - \gamma \theta + \frac{(1-\delta)^{2}}{1-\tau_{SS}^{w}} < 0,
\]

with \( \phi = \frac{(1-s)^{2}(1-\tau_{SS}^{w})^{2} - s(1-s)(1-\tau_{SS}^{w})^{2} + s^{2} + s\tau_{SS}^{w}(1-s)}{(1-s)^{2}(1-\tau_{SS}^{w})} \) and \( \gamma = \frac{(2-\delta)(1-s)(1-\tau_{SS}^{w})-2s(1-\delta) - (1-s)(1-\delta)\tau_{SS}^{w}}{(1-s)(1-\tau_{SS}^{w})} \).

In addition, \( P_{1}(\theta) \) has a root in \( \left( \frac{1}{1-\tau_{SS}^{w}}, \bar{\theta} \right) \), which is \( \theta_{1} = \frac{\gamma + \sqrt{\gamma^{2} - 4\phi}}{2\phi} \). And \( P_{1}(\theta) < 0 \) holds for all \( \theta \in \left( \frac{1}{1-\tau_{SS}^{w}}, \theta_{1} \right) \). When \( \delta > \frac{2s}{(1-\tau_{SS}^{w})(1-s)}, \theta_{1} < \bar{\theta} < \tilde{\theta} \) can hold for a set of properly chosen parameters. A numerical example is \( \tau_{SS}^{w} = 0.1, \delta = 1 \) and \( s = 0.3 \).

Following Cazzavillan and Pintus (2004), it is easy to show that \( \text{slope}_{\Delta}(0) > \text{slope}_{\Delta}(\bar{\sigma}) > \text{slope}_{\Delta}(\sigma_{H}) > \text{slope}_{\Delta}(\sigma_{T}) = 1. \)


It needs the following lemma.

**Lemma 2.** Let \( \frac{1}{1-\tau_{SS}^{w}} < \theta < \theta_{2} = \frac{\gamma + \sqrt{\gamma^{2} - 4\phi}}{2\phi} \), with \( \gamma = 2 - \frac{s}{(1-s)(1-\tau_{SS}^{w})} \) and assume that either \( R_{3} < \bar{R}_{3} = \frac{1}{1-\tau_{SS}^{w}} \), if \( \frac{1}{1-\tau_{SS}^{w}} < \theta < \theta_{1} \), or \( R_{3} < \bar{R}_{3} = \frac{2-\theta - \chi(\theta)}{1-\tau_{SS}^{w}} + \frac{\tau_{SS}^{w}}{1-\tau_{SS}^{w}} \), if \( \theta_{1} < \theta < \theta_{2} \).

Moreover, we assume that \( R_{1} > \bar{R}_{1} = \frac{\theta(1-\tau_{SS}^{w})-1}{1-\tau_{SS}^{w}} \). Then we have \( T_{0}(0) > 1 + D_{0}(0), D_{0}(0) < 1, 1 < T_{0}(0) < 2 \) and \( \text{slope}_{\Delta}(0) > \frac{1}{2T_{0}(0)} \). In other words, the origin \( (T_{0}(0), D_{0}(0)) \) lies outside the triangle ABC and the slope of the half-line \( \Delta(\sigma) \) is steeper than that of the line connecting the origin with the point C.
Proof. We require that $R_3 < \bar{R}_3 = \frac{1-\theta\chi(\theta)}{\theta\chi(\theta)}$ in order to get $D_0(0) < 1$. If $(T_0(0), D_0(0))$ lies outside the triangle ABC, it implies that $T_0(0) > 1 + D_0(0)$. This inequality holds iff

$$R_1 < \bar{R}_1 = \frac{\theta(1 - \tau_{w}^{NSS}) - 1}{\theta(1 + \chi(\theta)(1 + R_3)) - \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}}$$

with $R_3 > \bar{R}_3 = \frac{\theta - 1 - \chi(\theta)(1 + \chi(\theta))}{\theta(1 + \chi(\theta)) - \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}}$, $\bar{R}_1 > 0$ implies that $R_3 > \frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}}$. $T_0(0) < 2$ iff $R_1 > \bar{R}_1 = \frac{\theta(1 - \tau_{w}^{NSS}) - 1}{\theta(1 - \tau_{w}^{NSS}) + \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}}$. $\bar{R}_3$ is another upper bound on $R_3$. Notice that $\bar{R}_3(\theta)$ and $\bar{R}_3(\theta)$ are decreasing in $\theta$. To be accurate, $\bar{R}_3(\theta)$ goes down from $\left[1 - \chi\left((1 - \tau_{w}^{NSS})^{-1}\right)/(1 - \tau_{w}^{NSS})\right] / \left[\chi\left((1 - \tau_{w}^{NSS})^{-1}\right)/(1 - \tau_{w}^{NSS})\right]$ to $[1 - \theta_2\chi(\theta_2)] / \theta_2\chi(\theta_2)$, whereas $\bar{R}_3(\theta)$ goes down from $+\infty$ to $\frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}}$. In order to have a unique $\theta$ in the interval $(1 - \tau_{w}^{NSS})^{-1}, \theta_2)$ such that $\bar{R}_3 = \bar{R}_3$, we require that $\bar{R}_3(\theta_2) > \frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}}$, or, $\delta > \frac{\theta_2\chi(\theta_2)}{\theta_2} + \tau_{w}^{NSS}$, which can hold for small $\tau_{w}^{NSS}$. The unique $\theta$ satisfying $\bar{R}_3 = \bar{R}_3$, is the same $\theta_1$ obtained in lemma 1. As a result, one has $\bar{R}_3 > \bar{R}_3$, for $\theta$ in $(1 - \tau_{w}^{NSS})^{-1}, \theta_1)$, and $\bar{R}_3 < \bar{R}_3$, for $\theta$ in $(\theta_1, \theta_2)$. Put it differently, when $\theta$ is fixed in $(1 - \tau_{w}^{NSS})^{-1}, \theta_2)$, there is only one upper bound on $R_3$, which is either $\bar{R}_3$, when $\theta \in ((1 - \tau_{w}^{NSS})^{-1}, \theta_1)$, or $\bar{R}_3$, when $\theta \in (\theta_1, \theta_2)$.

Another condition is that $slope_\Delta(0) > (1 - D_0(0))/(2 - T_0(0))$. It implies that $R_1 > \bar{R}_1 = \frac{\theta(1 - \tau_{w}^{NSS}) - 1}{\theta(1 + \chi(\theta)(1 + R_3)) - \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}}$. $R_1 < \bar{R}_1$ implies that $R_3 > \frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}}$. $T_0(0) < 2$ iff $R_1 > \bar{R}_1 = \frac{\theta(1 - \tau_{w}^{NSS}) - 1}{\theta(1 - \tau_{w}^{NSS}) + \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}}$. $\bar{R}_3 < 1$ holds. Furthermore, $\bar{R}_1 > \bar{R}_1$ holds for $\theta \in ((1 - \tau_{w}^{NSS})^{-1}, \theta_2)$. The reason is that $\Sigma(\bar{R}_1 - \bar{R}_1) = \Sigma\left(2 - \theta - \frac{\chi(\theta)}{1 - \tau_{w}^{NSS}}\right) - \left(R_3 - \frac{\tau_{w}^{NSS}}{1 - \tau_{w}^{NSS}}\right)\left[\theta(1 - \tau_{w}^{NSS}) - 1\right]$. If the
Lemma 3. A decreasing function of $B$ as follows: $A! (1)$ with $s$ is an increasing function of $(D-2')$ as follows: $\lim_{s \to 0} \text{term is an increasing function of } (D-2')$ after we pin down the unique $0$ as long as $D_0(0) < 1$ and $R_3 < R_3$. It follows that $R_1 > R_1$ implies both $T_0(0) < 2$ and $\text{slope}_{\Delta}(0) > \frac{1-D_0(0)}{2-T_0(0)}$ as long as $D_0(0) < 1$ and $R_3 < R_3$ hold. ■

A.4. Proof of Proposition 4

If $(\overline{k}, \overline{s}) = (1, 1)$ is a normalized steady state of the dynamic system (25) and (26), we have the following: ($\overline{c}_1$ is the steady state of the first period consumption.)

$$A\rho (1) + 1 - \delta - g = u_2^{-1} \left( \frac{U_3'(1)}{\beta A\omega (1)} \right), \quad \text{(D-1')}$$

$$A\omega (1) - 1 = B(U'_1)^{-1} \left( \frac{BU_3'(1)}{A\omega (1)} \right) = \overline{c}_1. \quad \text{(D-2')}$$

If $g = \tau_k^{nss} A\rho (1)$ is not too large ($0 < \tau_k^{nss} < 1$ is the steady state capital income tax rate), $A\rho (1) + 1 - \delta - g > 0$ can hold. Since the LHS term of (D-2’) is positive, it implies that $A > 1/\omega (1)$. We rewrite (D–1’) as follows: $\beta A\omega (1) u_2[A\rho (1) + 1 - \delta - g] = U'_3 (1)$ and we find that the LHS term is an increasing function of $A$. In order to have a unique $A^*$ satisfying (D-1’), we require that $\beta A\omega (1) u_2[A\rho (1) + 1 - \delta - g]_{A=1/\omega (1)} < U'_3 (1)$. It is equivalent to $\beta u_2[\frac{\rho (1)}{A^1(1)} + 1 - \delta - g] < U'_3 (1)$. We can easily get $B^*$ from (D-2’) after we pin down the unique $A^*$ from (D-1’). In particular, we can rewrite (D-2’) as follows: $\frac{A\omega (1)-1}{B} U'_1(\frac{A\omega (1)-1}{B}) = \frac{A\omega (1)-1}{A\omega (1)} U'_3(1)$. It is easy to see that $\frac{A\omega (1)-1}{B} U'_1(\frac{A\omega (1)-1}{B})$ is a decreasing function of $B$. In order to have the unique $B^*$, we should impose the restriction:

$$\lim_{c \to 0} U'_1(c) < \frac{A^*\omega (1)-1}{A^*\omega (1)} U'_3(1).$$

A.5. Proof of Theorem 3

**Lemma 3.** Let $1 < \theta < \theta' = \frac{\tau^2 - 4\phi'(1-\delta)^2}{2\delta^2}$, where $\tau' \equiv \frac{1-s+(1-\delta)(1-2s)-(1-\delta)(1-\tau_k^{nss})}{1-s} \ast \phi' \equiv \frac{(1-s)^2-s(1-2s)(1-\tau_k^{nss})}{2\delta^2}$. Moreover, we assume that $R_1 > R_1$ and $R_3 < R_3 < R_3$, where $R_3 = \frac{\chi(\theta)(\theta-1)-\chi(\theta)(1-\tau_k^{nss})}{1+\chi(\theta)(1-\tau_k^{nss})}, \overline{R}_3 = \frac{1-(1-\tau_k^{nss})\chi(\theta)(1-\tau_k^{nss})}{(1-\tau_k^{nss})\chi(\theta)(1-\tau_k^{nss})},$ and $\overline{R}_1 = \frac{(\theta-1)(R_3-s\tau_k^{nss}/(1-s))}{(\theta)(R_3-s\tau_k^{nss}/(1-s))}. \ast \lim_{c \to 0} U'_1(c) = \chi(\theta).$ Then we have the following results: the origin $(T_0(0), D_0(0))$ lies inside the ABC
triangle and the half line \( \Delta(\sigma) \) intersects the interior of the segment \( BC \) at \( \sigma = 0 \) \((T_0(0) < 1 + D_0(0), D_0(0) < 1)\). Moreover, we have \( \text{slope}_\Delta(0) > \text{slope}_\Delta(\bar{\sigma}) > \text{slope}_\Delta(\sigma_H) > \text{slope}_\Delta(\sigma_T) = 1 \).

**Proof.** Similar to Cazzavillan and Pintus (2004), \( D_0(0) < 1 \) is satisfied iff \( 0 < R_3 < 1 - \frac{(1 - \tau_k^{\text{nss}} \mu) \chi(\theta) \theta}{(1 - \tau_k^{\text{nss}} \mu) \chi(\theta) \theta} \equiv \overline{R}_3 \), where \( \chi(\theta) = \frac{s\theta + (1 - s)(1 - \delta)}{(1 - s)\theta} \) and \( \mu = \frac{s}{(1 - s)\chi(\theta)} \). This \( \overline{R}_3 > 0 \) requires that \( \theta < \overline{\theta} \equiv \frac{(1 - s)\delta}{(1 - \tau_k^{\text{nss}} \mu)s} \).

Since \( \theta > 1 \), we know that \( s < \frac{\delta}{1 - \tau_k^{\text{nss}} \delta} < 1 \).

\[
1 + D_0(0) - T_0(0) > 0 \text{ is satisfied iff } R_1 > \overline{R}_1 \equiv \frac{(\theta - 1)[R_3 - s\tau_k^{\text{nss}}/(1 - s)]}{\chi(\theta)s(1 + R_3)(1 - \tau_k^{\text{nss}} \mu) - \chi(\theta)(\theta - 1)[1 + s\tau_k^{\text{nss}}/(1 - s)]} \text{ with } R_3 > \overline{R}_3 \equiv \frac{\chi(\theta) + (\theta - 1)[1 + s\tau_k^{\text{nss}}/(1 - s)]}{\chi(\theta)s(1 - \tau_k^{\text{nss}} \mu)} - 1. \text{ Since } R_1 < 1, \text{ we need that } \overline{R}_1 < 1, \text{ which is equivalent to }
\]

\[
R_3 > \overline{R}_3 = \frac{\chi(\theta) + (\theta - 1) - \chi(\theta) \theta (1 - \tau_k^{\text{nss}} \mu)}{1 + \chi(\theta) \theta (1 - \tau_k^{\text{nss}} \mu) - \theta},
\]

where \( 1 + \chi(\theta) \theta (1 - \tau_k^{\text{nss}} \mu) - \theta > 0 \). \( 1 + \chi(\theta) \theta (1 - \tau_k^{\text{nss}} \mu) - \theta > 0 \) holds iff \( \theta < \overline{\theta} \equiv \frac{(2 - \delta)(1 - s)}{1 - s - s[1 - \tau_k^{\text{nss}} \mu]} \).

It is easy to verify that if \( \delta > \frac{2s(1 - \tau_k^{\text{nss}})}{1 - s} \), the binding upper bound on \( \theta \) is \( \overline{\theta} \), as \( \overline{\theta} < \overline{\theta} \). Otherwise, if \( \delta < \frac{2s(1 - \tau_k^{\text{nss}})}{1 - s} \), the binding upper bound on \( \theta \) is \( \overline{\theta} \), as \( \overline{\theta} > \overline{\theta} \). In addition, \( \overline{R}_3 > \overline{R}_3 \) when \( D_0(0) < 1 \) is satisfied. Then we have that \( D_0(0) < 1 \) and \( T_0(0) < 1 + D_0(0) \) iff \( R_1 > \overline{R}_1 \) and \( \overline{R}_3 < R_3 < \overline{R}_3 \), provided that either \( \theta < \overline{\theta} \), when \( \delta < \frac{2s(1 - \tau_k^{\text{nss}})}{1 - s} \), or \( \theta < \overline{\theta} \), when \( \delta > \frac{2s(1 - \tau_k^{\text{nss}})}{1 - s} \). The inequality \( \overline{R}_3 < R_3 < \overline{R}_3 \) holds iff the polynomial holds.

\[
P_1(\theta) = \phi' \theta^2 - \Upsilon \theta + (1 - \delta)^2 < 0,
\]

with \( \phi' = \frac{(1 - s)^2 - s(1 - 2s)(1 - \tau_k^{\text{nss}})}{(1 - s)^2} \) and \( \Upsilon' = \frac{1 + (1 - \delta)(1 - 2s) - s(1 - \delta)(1 - \tau_k^{\text{nss}})}{1 - s} \). In addition, \( P_1(\theta) \) has a root in \( (1, \overline{\theta}) \), which is \( \theta_1' = \frac{\Upsilon' + [\Upsilon'^2 - 4\phi' (1 - \delta)^2]^{1/2}}{2\phi'} \). And \( P_1(\theta) < 0 \) holds for all \( \theta \in (1, \theta_1') \). When \( \delta > \frac{2s(1 - \tau_k^{\text{nss}})}{1 - s} \), \( 1 < \theta_1' < \overline{\theta} < \overline{\theta} \) can hold for a set of properly chosen parameters. A numerical example is \( \tau_k^{\text{nss}} = 0.2, \delta = 1 \) and \( s = 1/3 \).

Following Cazzavillan and Pintus (2004), it is easy to show that \( \text{slope}_\Delta(0) > \text{slope}_\Delta(\overline{\sigma}) > \text{slope}_\Delta(\sigma_H) > \text{slope}_\Delta(\sigma_T) = 1 \).
References


Tables and Figures

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<th>$\sigma_H$</th>
<th>$\tau_{w}^{NSS} = 0.1$</th>
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Table 1.

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Table 2. Numerical exercise: $\bar{\sigma}$ ($s = 1/3$, $\delta = 1$, $R_1 = 0.95$, and $R_3 = 0.82$).
Figure 1.

Figure 2.
Figure 3. $\theta_1$ (the vertical axis) is not a monotone function of $\tau_{w}^{NSS}$ (the horizontal axis).

Figure 4.